ALTERNATING CURRENTS

Alternating Currents

Their Elements Explained, and their Calculation Effected without the Use of Hyperbolic Functions

BY

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With Numerous Illustrations

NEW YORK

D. APPLETON AND COMPANY

1916

PRINTED IN GREAT BRITAIN
BY THE
DARIEN PRESS, EDINBURGH

PREFACE

In the Preface to the First Edition of my "Handbook of Electrical Testing" (published in 1876) I wrote as follows:—

"My aim has been not only to explain the practical application of formulæ given in existing text-books, but also to show by what processes they may be obtained. When an amateur in electrical science, I had experienced considerable difficulty in satisfying myself upon this point, and I know that others have been, and are, similarly in need of information and guidance. With the view, therefore, of doing something to aid students and those who may be concerned or interested in the inquiry, the present treatise is offered."

These words have again and again recurred to me in the course of my career, and I have several times been asked whether I could not bring out a treatise on the lines of the book above referred to, but dealing with "alternating" instead of with "continuous" currents. This is a task which my professional duties have, so far, prevented my undertaking. With the view, however, of doing *something* towards helping students, I have put together in the present handbook some notes which I had made from time to time dealing with certain problems which had interested me, and in reference to which I had been unable to find any sufficient explanations. The fact that a calculating machine

gives unerring results, and that in certain problems the application of "complex quantities" enables correct formulæ to be obtained, may be considered by some to be all-sufficient explanation. It has not, however, satisfied me, and I know that it has not satisfied many others.

It has been stated* that "problems in which the current has a continually changing value in consecutive sections of the line, due either to leakage between the two sides of the line or to the capacity of the line, can be accurately solved *only* by the use of hyperbolic trigonometry." I think that I have shown in the following pages that this is hardly a correct statement.

As far as possible the international symbols agreed upon by the International Electrotechnical Commission are used throughout the book. Also the symbols in the letterpress have been made to correspond in style with the lettering on the figures, a practice which, for no apparent reason, is seldom followed.

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BETCHWORTH,
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* See "Hyperbolic Functions and their application to Problems in Electrical Engineering," by J. H. Morecroft, Assist. Prof. of Electrical Engineering, Columbia University.

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ALTERNATING CURRENTS

CHAPTER I

GENERAL

Nomenclature.

Name of Unit.	Sign.	Name of Unit.	Sign.
Ampere	A	Farad	F
Milliampere	mA	Microfarad	μF
Volt	V	Henry	H
Ohm	O	Millihenry	mH
Megohm	MO		

1. If an electromotive force E (Fig. 1) produces a steady current I from **A** to **B** through a resistance r in

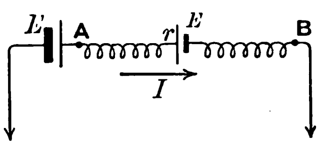

Fig. 1.

which a back electromotive force $\mathcal{E}$ exists, then we have—

$$I=\frac{E-\mathcal{E}}{r}.$$

If the electromotive force ${\scriptstyle E}$ be removed, and a resistance r_1 (Fig. 2) be substituted, r_1 being of such

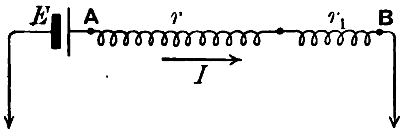

Fig. 2.

a value that the current flowing through r and r_1 from **A** to **B** is still I, then we have—

$$I = \frac{E}{r + r_1} = \frac{E - {\scriptstyle E}}{r},$$

therefore,

$$Er = Er - {\scriptstyle E}r + r_1(E - {\scriptstyle E}),$$

or,

$$r_1(E - {\scriptstyle E}) = {\scriptstyle E}r,$$

that is,

$$r_1 = r\frac{{\scriptstyle E}}{E - {\scriptstyle E}}.$$

For example, if $E = 20$**V**, ${\scriptstyle E} = 5$**V**, $r = 30$**O**, then,

$$I = \frac{20 - 5}{30} = \cdot 5\mathbf{A}.$$

But,

$$r_1 = 30\frac{5}{20 - 5} = 10\mathbf{O},$$

so that,

$$I = \frac{20}{30 + 10} = \cdot 5\mathbf{A}.$$

The effect of the back electromotive force of 5V is therefore equivalent to a resistance of 10O added to the circuit.

2. Suppose, to go a step further, the value of E were dependent upon the strength of the current flowing, that is to say, suppose we have—

$$E = Ik, \text{ or } k = \frac{E}{I},$$

where k is a constant, then we should have,

$$I = \frac{E - Ik}{r},$$

therefore,

$$Ir = E - Ik,$$

therefore,

$$I(r + k) = E,$$

or,

$$I = \frac{E}{r + k}; \qquad - \quad - \quad - \quad \text{(A)}$$

that is to say, in such a case the current produced by $\dfrac{E - E}{r}$ would be equivalent to the current produced by $\dfrac{E}{r + k}$.

3. In actual practice, however, we do not meet with cases in which a continuous current sets up a back electromotive force whose value is proportional to the current flowing. But the case is different where alternating currents are concerned, for if a current traversing a resistance which is inductive (that is to say, which is an

inductance) *changes* its value, then an electromotive force, due to this change, is set up in the inductance. The value of this electromotive force will be proportional to the *rate* of change taking place; thus, for example, if the current were to increase uniformly at the rate of 1**A** per second, *i.e.*, from, say, 10**A** to 11**A** during one second and from 11**A** to 12**A** during the next second, and so on, and if the value of the inductance is 1**H**, then the value of the electromotive force would be 1**V**. If the current were to increase at the rate of, say, 5**A** in two seconds—equivalent to 2·5**A** per second —and the value of the inductance were 3**H**, then the electromotive force set up would be—

$$E = 3 \times 2{\cdot}5 = 7{\cdot}5\mathbf{V}.$$

The electromotive force would continue only during the time the increase is taking place; if the increase ceased to continue and the current continued steady, the electromotive force would fall to zero. This fall to zero would also, of course, take place if the current ceased to flow. But the current may not increase at a uniform rate; for example, during one second it may increase from 10**A** to 11**A**, during the next second from 11**A** to 13**A**, during the third second from 13**A** to 16**A**, and so on. In this case, if the inductance were 1**H**, then during the first second the electromotive force set up would be 1**V**, during the second second it would be 2**V**, during the third second 3**V**, and so on.

If the current, instead of *in*creasing uniformly were to *de*crease uniformly, then an electromotive force would still be set up, but its direction would be reversed, *i.e.*, it would act *with* the decreasing current.

If the current alternately rises and falls, the back electromotive force will be alternately negative and positive, since on the *rise* of the current the force opposes the current, and on the *fall* of the current it acts with it.

4. When an alternating current follows—as is usually the case—a *sine* law, then the formula which gives the relation between I, E, r, and the inductance of r, is not that shown by (A) (p. 3), although it is of a similar general form; actually the magnitude of the voltage or electromotive force producing that alternating current will, if we assume that the angular change takes place at a uniform rate, vary directly as the sine of the angle of a complete period.

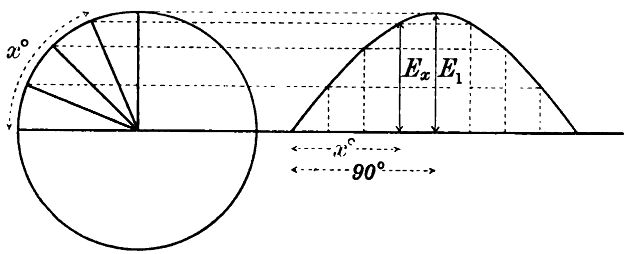

Fig. 3.

In Fig. 3 the left hand figure represents the angular change, and the right hand figure the corresponding sine wave form.

As the ordinates representing voltages are proportional to the sines of the angular distances, we have—

$$E_x = E_1 \sin x°,$$

where E_1 is the maximum value which the electro-

motive force reaches. This maximum value is, of course, attained when $x=90°$, for then $\sin x°=1$ (the greatest value a sine can have), *i.e.*,

$$E_x = E_1.$$

If, now, I_x be the current which will flow through a resistance r when the electromotive force has risen to E_x, then—

$$I_x = \frac{E_x}{r} = \frac{E_1 \sin x°}{r}.$$

5. In the simple case considered, r is assumed to possess *ohmic resistance* only, and the curve indicating the rise and fall of electromotive force is simple in construction. If, however, r possesses *inductance* in addition to ohmic resistance, then the conditions become modified, as explained in the next chapter.

CHAPTER II

(1)

ALTERNATING CURRENTS AND INDUCTANCE

6. If a resistance through which a current is flowing has inductance, and the current changes at a uniform rate, then an opposing electromotive force, due to the presence of the inductance, will be set up during the time the change is taking place. The value of the electromotive force set up at any moment will be directly proportional to (*a*) the value of the inductance, (*b*) to the *rate* of change of the current at that moment, and (*c*) to the current strength. If we call E the electromotive force, and L the coefficient of inductance, then—

$$E = L \times \text{rate of change of current.} \quad - \quad \text{(B)}$$

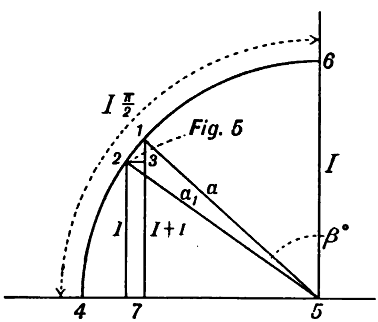

Fig. 4.

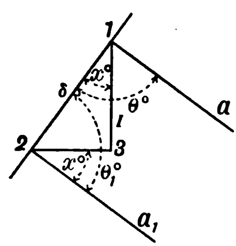

Fig. 5.

Let the current at any point 7 (Fig. 4) change from I to $I+\iota$ (ι being very small), in a very short time t; also let the number of periods per second be n; then the time taken by the current in changing from 0 to I (the maximum value) will be $\frac{1}{4n}$ seconds.

Now, as $I\frac{\pi}{2}$ represents the time corresponding to $\frac{1}{4n}$, the time t, corresponding to δ (Fig. 5), will be

$$t=\frac{1}{4n}\cdot\frac{\delta}{I\frac{\pi}{2}}=\frac{\delta}{I2\pi n}=\frac{\delta}{If},$$

if $f=2\pi n$.

Also, since ι is very small, the angle $\beta°$ (Fig. 4) is also very small, and consequently a and a_1 are practically parallel, so that the angles $\theta°$, $\theta_1°$ (Fig. 5) are both right angles; this being so, we have $\frac{\iota}{\delta}=\cos x°$, or $\iota=\delta\cos x°$.

But rate of change is $\frac{\iota}{t}$,

which equals

$$\delta\cos x°\div\frac{\delta}{If}=If\cos x°;$$

therefore, by substitution in (B), p. 7, we get

$$E=ILf\cos x°,$$

at any angle $x°$.

The greatest value which E can have is that which it has when $\cos x°$ is greatest, *i.e.*, when $x°=0°$, or $\cos x°=1$, in which case

$$E=ILf. \quad - \quad - \quad - \quad - \quad \text{(C)}$$

7. If, now, r (Fig. 6) be an inductive resistance (*inductance*) in a Wheatstone bridge, then if the circuit of the battery E be closed by depressing key k as shown, and if, when the key is kept down, the arms of the bridge be adjusted so as to obtain balance on the galvanometer, then on raising the key so that the battery circuit is opened, a discharge will take place from the inductive resistance, and a throw of the galvanometer needle will be produced; the direction of

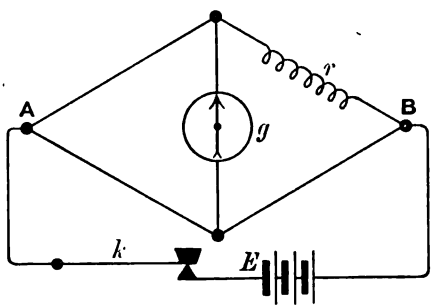

Fig. 6.

this throw will be found to indicate that the discharge current from the inductance is in the *same* direction as the charging current. If the key be now depressed, an equal but opposite throw to that just observed will be obtained, indicating that a discharge current is taking place from the inductance in the *opposite* direction to the charging current.

8. If we have a condenser (C) and battery (E), joined up as shown by Fig. 7 (p. 10), then on pressing down the key k, so as to short-circuit the battery, the condenser will discharge and give a throw on the galvanometer,

which throw will be in the *reverse* direction to the current which has charged the condenser; also on raising the key an equal but opposite throw will be obtained.

Precisely similar effects would be produced if a reversed current, either instantaneous, or rising and falling

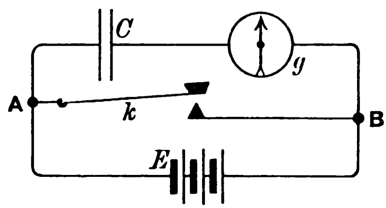

Fig. 7.

quickly and regularly, were in either case (Figs. 6 and 7) sent between the points A and B. If the effects were recorded on an "Oscillograph," instead of being simply the throws on a galvanometer, it would be found that the curves would be similar in the two cases.

9. In Fig. 8 let A be the sine curve of a resultant[1] electromotive force generating a current through a resistance r possessing an inductance L, then the current will set up an opposing electromotive force E, the value of which will vary according to a sine law, since it will be directly proportional to the variation of current, which current will vary directly as the resultant generating electromotive force. The curve B of E will not, however, coincide with the curve of the generating force, since it is obvious that when the latter is increasing at its greatest rate, which will be at the point

[1] Electromotive force diminished by effect of inductance.

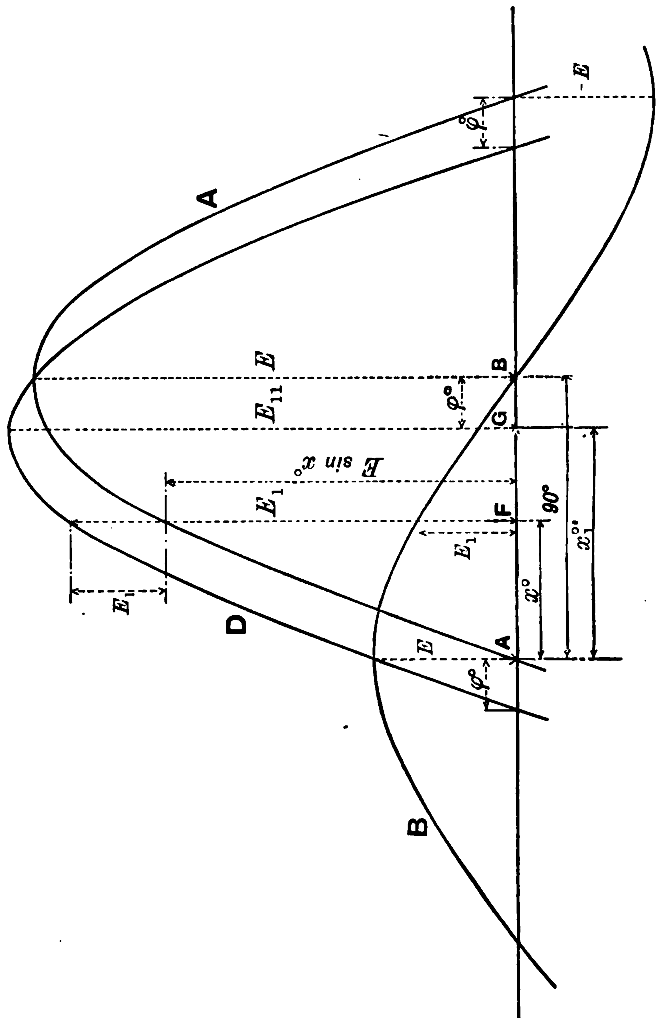

Fig. 8.

A, then at that point E will be at its maximum value; whilst, on the other hand, when the generating force attains its maximum value E at **B**, then this force will at that point be at zero as regards its rate of increase; consequently E will have a zero value at **B**—in other words, the opposing electromotive force curve will be 90° out of phase with the generating resultant electromotive force. The generating electromotive force, then, at any point **F** will be

$$E \sin x^\circ,$$

whilst the opposing electromotive force E_1, at that point, will be

$$E_1 = E \sin (90^\circ - x^\circ) = E \cos x^\circ.$$

To determine the value of the "impressed" electromotive force E_{11}, which will produce the resultant E, we assume that the ordinates of the sine curve **D**, which represents the rise and fall of E_{11}, is compounded of the ordinates of the curves **A** and **B**.

Now,

$$E_1 = E \sin x^\circ + E_1,$$

therefore,

$$E_1 = E \sin x^\circ + E \cos x^\circ \qquad - \qquad - \quad \text{(D)}$$

This curve **D** will attain its maximum value, E_{11}, at a point **G**, intermediate between **A** and **B**.

Angle of Lag.

10. The distance between E and E_{11} is called the "angle of lag," that is to say, the maximum value E

is attained before (*i.e.*, in advance of) the maximum value E_{11} is reached.

11. In order to find the magnitude of the maximum value E_{11} in terms of E and ${\scriptstyle E}$, we must find what value of x makes E_1 a maximum.

Since

$$E_1 = E \sin x^\circ + {\scriptstyle E} \cos x^\circ,$$

we have

$$\frac{dE_1}{dx} = E \cos x^\circ - {\scriptstyle E} \sin x^\circ = 0$$

at a maximum, so that

$$E \cos x^\circ = {\scriptstyle E} \sin x^\circ.$$

[This result may be obtained, without the use of the calculus, in the following way :—

$$E \sin x^\circ + {\scriptstyle E} \cos x^\circ = E\left(\sin x^\circ + \frac{{\scriptstyle E}}{E} \cos x^\circ\right).$$

Let

$$\frac{{\scriptstyle E}}{E} = \frac{\cos \gamma^\circ}{\sin \gamma^\circ},$$

then the expression becomes

$$E\left(\sin x^\circ + \frac{\cos \gamma^\circ}{\sin \gamma^\circ} \cos x^\circ\right)$$

$$= \frac{E}{\sin \gamma^\circ} (\sin x^\circ \sin \gamma^\circ + \cos x^\circ \cos \gamma^\circ)$$

$$= \frac{E}{\sin \gamma^\circ} (\cos (x^\circ - \gamma^\circ)),$$

which is obviously a maximum when $\cos(x^\circ - \gamma^\circ)$ is a maximum, *i.e.*, when $\cos(x^\circ - \gamma^\circ) = 1$, or when $x^\circ = \gamma^\circ$ (since $\cos 0^\circ = 1$); so that

$$\frac{E}{E} = \frac{\cos x^\circ}{\sin x^\circ},$$

or,

$$E \cos x^\circ = E \sin x^\circ].$$

12. Now,

$$E^2 \cos^2 x^\circ = E^2 \sin^2 x^\circ,$$

or,

$$E^2 (1 - \sin^2 x^\circ) = E^2 \sin^2 x^\circ,$$

or,

$$E^2 - E^2 \sin^2 x^\circ = E^2 \sin^2 x^\circ;$$

therefore,

$$\sin^2 x^\circ (E^2 + E^2) = E^2,$$

or,

$$\sin x^\circ = \frac{E}{\sqrt{E^2 + E^2}};$$

similarly, since $\sin^2 x^\circ = 1 - \cos^2 x^\circ$, $\cos x^\circ = \dfrac{E}{\sqrt{E^2 + E^2}}$,

therefore,

$$E \sin x^\circ + E \cos x^\circ = \frac{E^2}{\sqrt{E^2 + E^2}} + \frac{E^2}{\sqrt{E^2 + E^2}}$$

$$= \frac{E^2 + E^2}{\sqrt{E^2 + E^2}} = \sqrt{E^2 + E^2} = E_{11}$$ (the maximum value of E_1), or

$$E_{11}^{\,2} = E^2 + E^2.$$

13. If, now, we have the electromotive force E_{11} sending a current I through a resistance r, in which there is an inductance L, then E_{11} becomes reduced to a resultant E, and the current which will flow will be

$$I = \frac{E}{r}.$$

But from equation (C), page 8, we see that

$$E^2 = I^2L^2f^2;$$

and since

$$I = \frac{E}{r},$$

therefore,

$$I^2r^2 = E^2;$$

from which, since $E_{11}{}^2 = E^2 + E^2$, we get

$$E_{11}{}^2 = I^2r^2 + I^2L^2f^2,$$

or

$$I^2(r^2 + L^2f^2) = E_{11}{}^2,$$

that is

$$I = \frac{E_{11}}{\sqrt{r^2 + L^2f^2}}. \qquad - \quad - \quad - \quad \text{(E)}$$

This equation shows that the maximum effect of the

inductance is equivalent to increasing the value of r to $\sqrt{r^2+L^2f^2}$. It follows, therefore, that if an alternating electromotive force of E**V** will send a current of I**A** through a non-inductive resistance of r**O**, then if r possesses an inductance of L**H**, it will be necessary to have an electromotive force of E_{11}**V** in order to send the same current, I, through r.

For example—

$$\text{If } E=80\mathbf{V},\quad r=20\mathbf{O},$$

then when r has no inductance,

$$I=\frac{80}{20}=4\mathbf{A};$$

but if there is inductance in r such that $Lf=4$, then r becomes

$$\sqrt{20^2+4^2}=\sqrt{416}=20{\cdot}396.$$

The voltage, therefore, which would in this case be required to produce a current of 4 **A** would be

$$4\times 20{\cdot}396=81{\cdot}584\mathbf{V}.$$

14. It may be noted that since

$$E=ILf$$

and

$$I = \frac{E}{r},$$

therefore,

$$E = E\frac{Lf}{r}$$

and

$$E_{11} = \sqrt{E^2 + E^2\frac{L^2f^2}{r^2}} = \frac{E}{r}\sqrt{r^2 + L^2f^2}.$$

Mechanical Analogy

15. The expression $\sqrt{r^2 + L^2f^2}$ is precisely similar to that giving the resultant r_1, of two forces r and Lf acting at right angles to each other, as shown by Fig. 9,

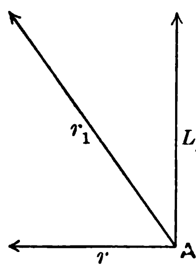

Fig. 9.

and for purposes of mathematical analysis the quantities may conveniently be so regarded.[1]

[1] It is often stated in dealing with problems of the kind in relation to alternating currents that Lf *is* (*i.e.*, is "actually") at right angles to r, a statement which is misleading as such.

(2)

ALTERNATING CURRENTS AND CAPACITY

16. It has been explained (p. 9, § 8) that capacity acts similarly to inductance, but in an inverse sense; the capacity curves, therefore, contrasted with those shown in Fig. 8 (p. 11), will be as indicated by Fig. 10 (on opposite page), and equation (D), p. 12, becomes

$$E_1 = E \sin x^\circ - E \cos x^\circ. \qquad \text{(F)}$$

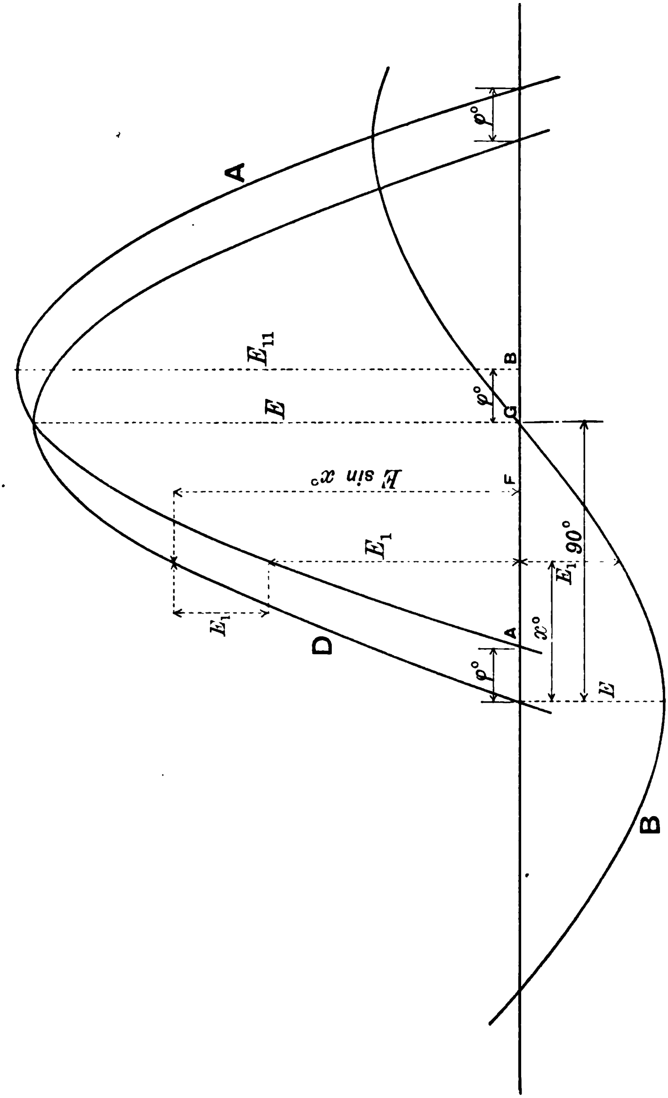

Fig. 10.

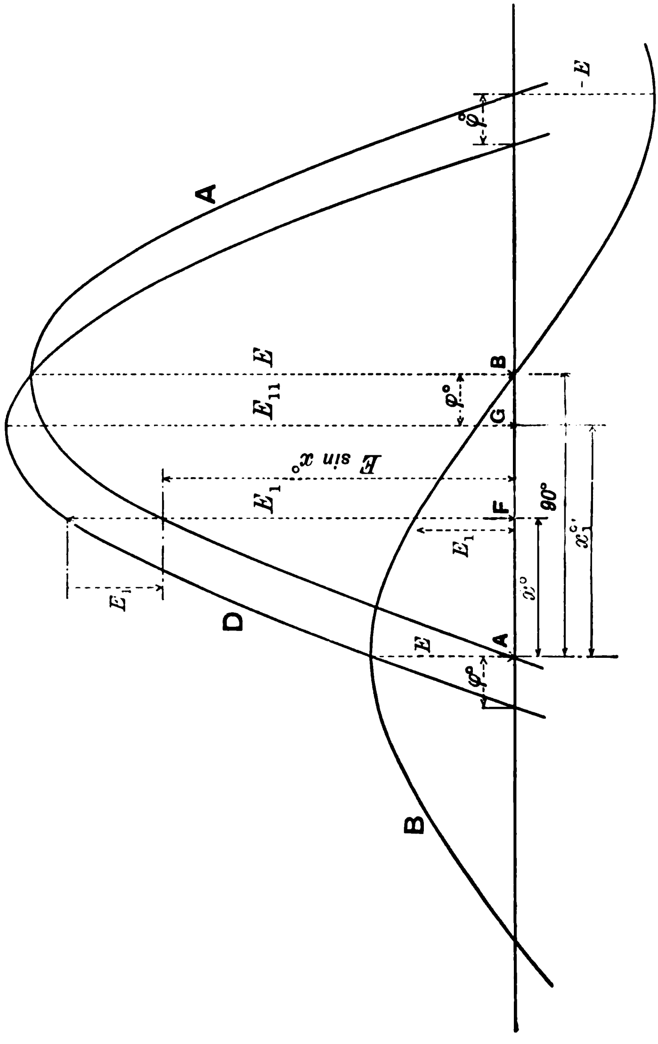

Fig. 8.

and x_1° is the value of x° when E_1 reaches a maximum, under which condition,

$$E_1 \cos x_1^\circ = E \sin x_1^\circ \text{ (p. 13)},$$

or,

$$\frac{E}{E} = \frac{\cos x_1^\circ}{\sin x_1^\circ} = \cot x_1^\circ = \tan \phi^\circ,$$

i.e.,

$$\tan \phi^\circ = \frac{E}{E};$$

but (p. 17),

$$E = E\frac{Lf}{r},$$

so that,

$$\tan \phi^\circ = \frac{Lf}{r}.$$

For example, taking the values of r and Lf given in the example, p. 16, viz., $r = 200$, $Lf = 4$, we have,

$$\tan \phi^\circ = \frac{4}{20} = \cdot 02 = \tan 11^\circ\ 18'.$$

18. Referring to Fig. 8 (p. 22), in which there are two curves **A** and **D**, the former without and the latter with inductance, then in order to compare the ordinate of **A** at **F** with the ordinate of **D** at the same point **F**, we must in the first case take the abscissa to be x°, and in the second case to be $\phi^\circ + x^\circ$; or if we wish to compare the ordinates of two curves, one of which has a lag of ϕ_1°, and the other of ϕ_2, then the abscissæ must be $\phi_1^\circ + x^\circ$ and $\phi_2^\circ + x^\circ$, respectively.

19. If, now, I_1 be a current sent by an impressed electromotive force E_{11} through an inductive resistance whose maximum value is, as has been shown, $\sqrt{r^2+L^2f^2}$, then (referring to Fig. 8) the value of this current will be—

$$I_1=E_{11}\frac{\sin(\phi^\circ+x^\circ)}{\sqrt{r^2+L^2f^2}}.$$

But

$$\sin(\phi^\circ+x^\circ)=\sin\phi^\circ\cos x^\circ+\cos\phi^\circ\sin x^\circ;$$

also

$$\tan\phi^\circ=\frac{Lf}{r}=\frac{\sin\phi^\circ}{\cos\phi^\circ},$$

therefore

$$\frac{L^2f^2}{r^2}=\frac{\sin^2\phi^\circ}{1-\sin^2\phi^\circ},$$

or

$$\sin^2\phi^\circ\, r^2=L^2f^2-L^2f^2\sin^2\phi^\circ,$$

therefore

$$\sin^2\phi^\circ\,(r^2+L^2f^2)=L^2f^2,$$

or

$$\sin\phi^\circ=\frac{Lf}{\sqrt{r^2+L^2f^2}};$$

similarly

$$\cos\phi^\circ=\frac{r}{\sqrt{r^2+{}^2Lf^2}},$$

therefore,

$$\sin(\phi^\circ + x^\circ) = \frac{Lf}{\sqrt{r^2+L^2f^2}} \cos x^\circ + \frac{r}{\sqrt{r^2+L^2f^2}} \sin x^\circ$$

$$= \frac{r \sin x^\circ + Lf \cos x^\circ}{\sqrt{r^2+L^2f^2}},$$

or,

$$I_1 = E_{11} \frac{\dfrac{r \sin x^\circ + Lf \cos x^\circ}{\sqrt{r^2+L^2f^2}}}{\sqrt{r^2+L^2f^2}} = E_{11} \frac{r \sin x^\circ + Lf \cos x^\circ}{r^2+L^2f^2}$$

$$= \frac{E_{11}}{\dfrac{r^2+L^2f^2}{r \sin x^\circ + Lf \cos x^\circ}} \quad - \quad - \quad - \quad \text{(G)}$$

i.e.,

$$\frac{r^2+L^2f^2}{r \sin x^\circ + Lf \cos x^\circ} \quad - \quad - \quad - \quad \text{(H)}$$

is the resistance.

The conditions for I_1 being a maximum, it may be noted, will be given when $r \sin x^\circ + Lf \cos x^\circ$ is a maximum, which will be the case, as the investigation on p. 13 shows, when it equals $\sqrt{r^2+L^2f^2}$, so that

$$I_1 = E_{11} \frac{\sqrt{r^2+L^2f^2}}{r^2+L^2f^2} = \frac{E_{11}}{\sqrt{r^2+L^2f^2}},$$

as was also shown on p. 15.

CHAPTER IV

JOINT RESISTANCES

Joint Resistance of Two Inductances in Series

20. In this case we do not require to consider the question of the phase of each resistance and its inductance, separately ; the case is that of a total ohmic

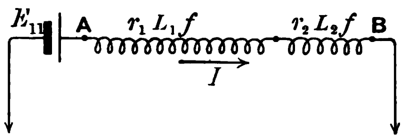

Fig. 11.

resistance r_1+r_2 between **A** and **B** having a total inductance of L_1f+L_2f, so that the maximum value which the combination will have will be

$$\sqrt{(r_1+r_2)^2+(L_1f+L_2f)^2}, \quad - \quad - \quad (a)$$

and not

$$\sqrt{r_1^2+L_1^2f^2}+\sqrt{r_2^2+L_1^2f^2}, \quad - \quad - \quad (b)$$

as might at first sight have been thought to have been the case.

Mechanical Analogy

21. In order to understand the meaning of (a) as contrasted with (b), let us take the case of two mechanical forces at right angles to each other, acting on a point, as in Fig. 12.

If a and b represent the two forces acting on the point **A**, then, as is well known, r_1 will be their resultant. This resultant could, if desired, be resolved in a different direction to that shown; it could, for example, be resolved in the direction 1, or in the

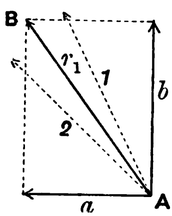

Fig. 12.

direction 2, but in either case the values of r_1 so resolved would be less than the value in the direction **AB**. In other words, the resultant r_1 in the direction **AB** is the *maximum* value that it can have (precisely as in the case of equation $\sqrt{E^2+E^2}=E_{11}$, p. 14), and, as is well known, under these conditions we have

$$r_1=\sqrt{a^2+b^2}.$$

If, now, we have a second set of forces c and d (Fig. 14, p. 29) acting upon **A**, then in this case we have a maximum resultant r_2 of the value

$$r_2=\sqrt{c^2+d^2}.$$

But if we desire to determine the maximum joint value, R, of r_1 (Fig. 14) and r_2 (Fig. 15) upon **A**, we should not obtain this by adding r_1 to r_2 (*i.e.*, R would not be equal to $\sqrt{a^2+b^2}+\sqrt{c^2+d^2}$), for r_1 and r_2 do not act in the same direction. It is obvious, however, that

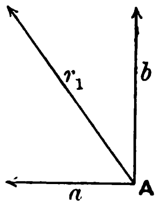

Fig. 13.

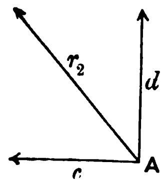

Fig. 14.

since c acts in the same direction as a, and d in the same direction as b, we can, as shown by Fig. 15, add a to c, and b to d, hence R will be given by the equation

$$R=\sqrt{(a+c)^2+(b+d)^2},$$

which is similar to (a), p. 27.

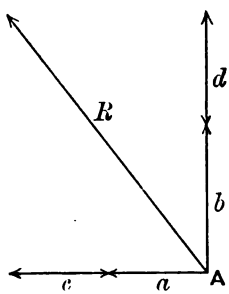

Fig. 15.

We see, then, how direction as well as magnitude

require to be taken into consideration in the case of the addition of forces.

Joint Resistance of Two Inductances in Parallel

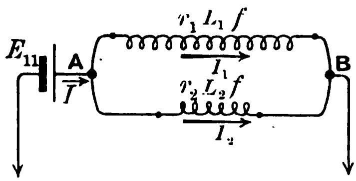

Fig. 16.

22. In Fig. 16, r_1 and r_2 are two resistances between **A** and **B**—both inductive—joined in parallel, then (p. 25)

$$I_1 = E_{11}\,\frac{r_1 \sin x^\circ + L_1 f \cos x^\circ}{r_1^2 + L_1^2 f^2},$$

also,

$$I_2 = E_{11}\,\frac{r_2 \sin x^\circ + L_2 f \cos x^\circ}{r_2^2 + L_2^2 f^2}$$

so that

$$I = I_1 + I_2 = E_{11}\left[\frac{r_1 \sin x^\circ + L_1 f \cos x^\circ}{r_1^2 + L_1^2 f^2} + \frac{r_2 \sin x^\circ + L_1 f \cos x^\circ}{r_2^2 + L_2^2 f^2}\right]$$

$$= E_{11}\,\frac{(r_2^2 + L_2^2 f^2)(r_1 \sin x^\circ + L_1 f \cos x^\circ) + (r_1^2 + L_1^2 f^2)(r_2 \sin x^\circ + L_2 f \cos x^\circ)}{(r_1^2 + L_1^2 f^2)(r_2^2 + L_2^2 f^2)}$$

$$= E_{11}\,\frac{[r_1(r_2^2 + L_2^2 f^2) + r_2(r_1^2 + L_2^2 f^2)] \sin x^\circ + [L_1 f(r_2^2 + L_2^2 f^2) + L_2 f(r_1^2 + L_1^2 f^2)] \cos x^\circ}{(r_1^2 + L_1^2 f^2)(r_2^2 + L_2^2 f^2)}. \quad \text{(I)}$$

From pp. 13-14 we can see that I will be a maximum when

$$I=E_{11}\sqrt{\frac{[r_1(r_2^2+L_2^2f^2)+r_2(r_1^2+L_1^2f^2)]^2+[L_1f(r_2^2+L_2^2f^2)+L_2f(r_1^2+L_1^2f^2)]^2}{[(r_1^2+L_1^2f^2)(r_2^2+L_2^2f^2)]^2}}.$$

Let the quantity under the square root be written

$$\frac{[a(b^2+d^2)+b(a^2+c^2)]^2+[c(b^2+d^2)+d^2(a^2+c^2)]^2}{[(a^2+c^2)(b^2+d^2)]^2};$$

this equals

$$\frac{\begin{array}{c}a^2(b^2+d^2)^2+b^2(a^2+c^2)^2+2\,ab(b^2+d^2)(a^2+c^2)+c^2(b^2+d^2)^2+\\ d^2(a^2+c^2)^2+2\,cd(b^2+d^2)(a^2+c^2)\end{array}}{[(a^2+c^2)(b^2+d^2)]^2}$$

$$=\frac{(a^2+c^2)^2(b^2+d^2)+(b^2+d^2)^2(a^2+c^2)+2\,(ab+cd)(a^2+c^2)(b^2+d^2)}{[(a^2+c^2)(b^2+d^2)]^2}$$

$$=\frac{a^2+c^2+b^2+d^2+2\,(ab+cd)}{(a^2+c^2)(b^2+d^2)}$$

$$=\frac{(a+b)^2+(c+d)^2}{(a^2+c^2)(b^2+d^2)}=\frac{(r_1+r_2)^2+(L_1f+L_2f)^2}{(r_1^2+L_1^2f^2)(r_2^2+L_2^2f^2)};$$

so that

$$I=\frac{E_{11}}{\sqrt{\frac{(r_1^2+L_1^2f^2)(r_2^2+L_2^2f^2)}{(r_1+r_2)^2+(L_1f+L_2f)^2}}}.$$

23. If we put L_1 and L_2 each equal to 0—*i.e.*, if we make the resistances non-inductive, then

$$I=\frac{E_{11}}{\frac{r_1r_2}{r_1+r_2}},$$

the ordinary formula for two ohmic resistances in parallel.

Individual Values of I_1 and I_2

24. If we require to know the individual values of I_1 and I_2 under the conditions where I_1+I_2 are a maximum, then from (I), p. 28, we know that for a maximum

$$[r_1(r_2^2+L_2^2f^2)+r_2(r_1^2+L_1^2f^2)] \cos x^\circ = [L_1(r_2^2+L_2^2f^2)+L_2(r_1^2+L_1^2f^2)] \sin x^\circ,$$

or, say,

$$A \cos x^\circ = B \sin x^\circ;$$

therefore,

$$A^2 \cos^2 x^\circ = B^2 \sin^2 x^\circ,$$

or

$$A^2(1-\sin^2 x^\circ) = B^2 \sin^2 x^\circ,$$

or

$$A^2 = A^2 \sin^2 x^\circ + B^2 \sin^2 x^\circ,$$

therefore,

$$\sin^2 x^\circ = \frac{A^2}{A^2+B^2},$$

or

$$\sin x^\circ = \frac{A}{\sqrt{A^2+B^2}};$$

similarly

$$\cos x^\circ = \frac{B}{\sqrt{A^2+B^2}}$$

from this we get (see equations, p. 30)

$$I_1 = E_{11} \frac{r_1 \frac{A}{\sqrt{A^2+B^2}} + L_1 f \frac{B}{\sqrt{A^2+B^2}}}{r_1^2+L_1^2f^2} = \frac{r_1 A + L_1 f B}{\sqrt{A^2+B^2}(r_1^2+L^2f^2)},$$

and

$$I_2 = E_{11}\frac{r_2\frac{A}{\sqrt{A^2+B^2}} + L_2 f\frac{B}{\sqrt{A^2+B^2}}}{r_2^2+L_2^2f^2} = \frac{r_2A+L_2f\,B}{\sqrt{A^2+B^2}(r_2^2+L_2^2f^2)};$$

substituting the values of A and B, these equations become

$$I_1 = \frac{E_{11}}{\sqrt{\frac{(r_1^2+L_1^2f^2)(r_2^2+L_1^2f^2)[(r_1+r_2)^2+(L_1f+L_2f)^2]}{[r_2(r_1+r_2)+L_2f(L_1f+L_2f)]^2}}},$$

and

$$I_2 = \frac{E_{11}}{\sqrt{\frac{(r_1^2+L_2^2f^2)(r_2^2+L_2^2f^2)[(r_1+r_2)^2+(L_1f+L_2f)^2]}{[r_1(r_1+r_2)+L_1f(L_1f+L_2f)]^2}}};$$

also, we have

$$\frac{I_1}{I_2} = \frac{r_2(r_1+r_2)+L_2f(L_1f+L_2f)}{r_1(r_1+r_2)+L_1f(L_1f+L_2f)}.$$

CHAPTER V

ELEMENTARY COMBINATIONS OF RESISTANCE, INDUCTANCE, AND CAPACITY

Resistance and Inductance in Series

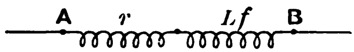

Fig. 17.

25. The value of this combination, Fig. 17, has been shown, p. 25, to be

$$\frac{r^2 + L^2 f^2}{r \sin x^\circ + Lf \cos x^\circ}, \qquad - \quad - \quad \text{(H)}$$

and at a maximum

$$\sqrt{r^2 + L^2 f^2}.$$

26. Although, strictly speaking, inductance cannot be considered as existing apart from resistance, *i.e.*, it is impossible practically to have an inductance without ohmic resistance (unlike the case of capacity, *i.e.*, of a condenser), yet for the purpose of mathematical analysis it may be considered as existing independent of the same.

Resistance and Inductance in Parallel

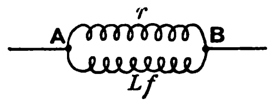

Fig. 18.

27. The expression for this combination we get from (I), p. 30, viz.:

$$I = E_{11}\frac{[r_1(r_2^2+L_2^2f^2)+r_2(r_1^2+L_2^2f^2)]\sin x^\circ+[L_1f(r_2^2+L_2^2f^2)+L_2f(r_1^2+L_1^2f^2)]\cos x^\circ}{(r_1^2+L_1^2f^2)(r_2^2+L_2^2f^2)},$$

by putting in this equation, $L=0$ and $r_2=0$; so that

$$I = E_{11}\frac{r_1 L_2^2 f^2 \sin x^\circ + L_2 f\, r_1^2 \cos x^\circ}{r_1^2 L_2^2 f^2}$$

$$=\frac{E_{11}}{\dfrac{r_1 L_2 f}{r_1 \cos x^\circ + L_2 f \sin x^\circ}};$$

or putting r for r_1, and L for L_2, the resistance will be

$$\frac{r\,Lf}{r\cos x^\circ + Lf\sin x^\circ},$$

which, at a maximum, equals

$$\frac{r\,Lf}{\sqrt{r^2+L^2f^2}}=\frac{1}{\sqrt{\dfrac{1}{r^2}+\dfrac{1}{L^2f^2}}}.$$

Resistance and Capacity in Series

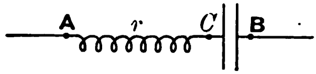

Fig. 19.

28. Referring to equation (G), p. 25, and equation (F), p. 18 (as compared with equation (D), p. 12), then, since capacity is inverse in its effect to inductance (*i.e.*, the greater the capacity the more freely does a current flow into it), we can see that equation (G) becomes

$$I_1 = \frac{E_{11}}{\dfrac{r^2 + \dfrac{1}{C^2 f^2},}{r \sin x^\circ - \dfrac{1}{Cf} \cos x^\circ}};$$

that is to say, the resistance of the combination is

$$\frac{r^2 + \dfrac{1}{C^2 f^2},}{r \sin x^\circ - \dfrac{1}{Cf} \cos x^\circ}.$$

29. If there is no ohmic resistance in circuit with the capacity, then putting $r = 0$, we get

$$I_1 = \frac{E_{11}}{\dfrac{\dfrac{1}{C^2 f^2}}{-\dfrac{1}{Cf} \cos x^\circ}} = E_{11}(-Cf \cos x^\circ). \qquad \text{(J)}$$

30. For a maximum the resistance of the combination will be—

$$\frac{r^2+\frac{1}{C^2f^2}}{\sqrt{r^2+\frac{1}{C^2f^2}}} = \sqrt{r^2+\frac{1}{C^2f^2}},^*$$

Resistance and Capacity in Parallel

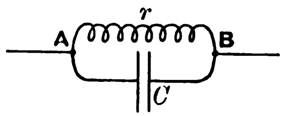

Fig. 20.

31. This is the condition we have in the case of the core of a submarine cable, r being in this case the 'insulation resistance."

If now I_2 be the current through a non-inductive resistance, r, then

* We cannot obtain this maximum value of $r \sin x^\circ - \frac{1}{Cf} \cos x^\circ$ (p. 37) by direct differentiation of the latter expression, for, as will be seen from Fig. 10, p. 19, x° is greater than 90°, being actually equal to $90^\circ+\phi^\circ$; if we insert this value $(90^\circ+\phi^\circ)$ in the expression $r \sin x^\circ - \frac{1}{Cf} \cos x^\circ$, the latter becomes

$$r \sin (90^\circ+\phi^\circ) - \frac{1}{Cf} \cos (90^\circ+\phi^\circ),$$

which equals

$$r \cos \phi^\circ + \frac{1}{Cf} \sin \phi^\circ,$$

which, as can be seen from the investigation in pp. 13-15, has as a maximum, the value

$$\sqrt{r^2+\frac{1}{C^2f^2}}.$$

$$I_2 = E_{11}\frac{\sin x^\circ}{r};$$

therefore the current I which will pass through the non-inductive resistance and the capacity (see equation (J), p. 37) will be

$$I = I_1 + I_2 = E_{11}(-Cf_1 \cos x^\circ) + E_{11}\frac{\sin x^\circ}{r}$$

$$= \frac{E_{11}}{\dfrac{1}{\dfrac{1}{r}\sin x^\circ - Cf\cos x^\circ}},$$

i.e., the joint resistance is

$$\frac{1}{\dfrac{1}{r}\sin x^\circ - Cf\cos x^\circ};$$

put $x^\circ = 95$

or if we put leakance, s, for insulation resistance, then the joint resistance is

$$\frac{1}{s\sin x^\circ - Cf\cos x^\circ} \quad - \quad - \quad - \quad \text{(K)}$$

32. For a maximum, the resistance will be (see note*, p. 38)—

$$\frac{1}{\sqrt{s^2 + C^2f^2}}.$$

CHAPTER VI

TELEPHONIC TRANSMISSION

ATTENUATION EQUATION

33. IF we consider the case of a steady current flowing through a cable, then the measure of the efficiency of the cable will be the amount by which the transmitted current has become reduced or "attenuated" when it reaches the further end; this "attenuation" is obviously the ratio of the transmitted (I_s) to the received (I_r) currents, *i.e.*, the ratio

$$\frac{I_s}{I_r}.$$

If $I_r = I_s$, *i.e.*, if there is no loss or attenuation, then

$$\frac{I_s}{I_r} = 1;$$

if

$$I_r = \frac{I_s}{2}, \text{ then}$$

$$\frac{I_s}{I_r} = \frac{I_s}{\frac{I_s}{2}} = 2,$$

and so on.

6

We see, then, that the greater the value of $\frac{I_s}{I_r}$ the greater is the attenuation and the less the efficiency of the cable for transmission purposes.

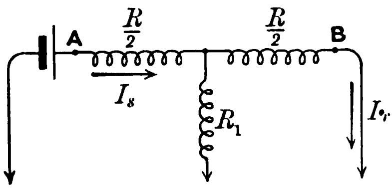

Fig. 21.

· 34. Let now **A B** be a cable having a total conductor resistance R, and a resultant insulation resistance R_1 at the centre of the cable, then since

$$I_r = I_s \frac{R_1}{R_1 + \frac{R}{2}},$$

therefore

$$\frac{I_s}{I_r} = \frac{R_1 + \frac{R}{2}}{R_1} = 1 + \tfrac{1}{2} \cdot \frac{R}{R_1}. \qquad - \qquad - \quad \text{(L)}$$

If, for example, $R = 360$, $R_1 = 1000$, then

$$\frac{I_s}{I_r} = 1 + \tfrac{1}{2} \cdot \frac{360}{1000} = 1{\cdot}180;$$

i.e., the cable efficiency (which is, of course, the reciprocal of the attenuation) is

$$\frac{1}{1 \cdot 180} = \cdot 847,$$

or 15·3 (*i.e.*, 100 − 84·7) per cent. less than that of a cable which had an infinite insulation resistance, *i.e.*, which would transmit the whole of the current sent.

35. If we substitute resistance (r) per unit length for total conductor resistance, and leakance (s) per unit length for total insulation resistance, then if l = length, we get

$$\frac{R}{R_1} = l^2 rs,$$

therefore

$$\frac{I_s}{I_r} = 1 + \tfrac{1}{2} l^2 rs. \qquad \text{(M)}$$

36. The foregoing formula gives the value of $\frac{I_s}{I_r}$ approximately only; the accurate formula is

$$\frac{I_s}{I_r} = \frac{(\sigma\frac{m}{r} + 1)e^{ml} + (\sigma\frac{m}{r} - 1)e^{-ml}}{2}, \qquad \text{(N)}$$

in which σ is a resistance at the receiving end of the cable, l the length of the cable, r the conductor resistance per unit length, $m = \sqrt{rs}$, s being the leakance per

unit length (or $m=\sqrt{\frac{r}{i}}$, if i is the insulation resistance). This formula is arrived at as follows:—

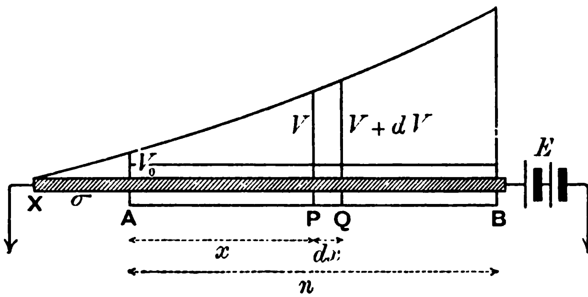

Fig. 22.

* Let **AB** (Fig. 22) be a cable of any length, connected to a battery E, as shown, and having its further end to earth through a resistance σ. By putting $\sigma=0$ the end of the cable will be direct to earth, and by putting $\sigma=\infty$, it will be insulated.

Let the length **AB** $=n$;
" " **AP** $=x$;
" " **PQ** $=dx$.

Let the potential at **A** $=V_0$;
" " **P** $=V$;
" " **Q** $=V+dV$.

* See "On the Leakage of Submarine Cables," by A. B. Kempe, B.A., *Journal of the Society of Telegraph-Engineers*, Vol. IV., p. 90 (1875); and "Handbook of Electrical Testing" (H. R. Kempe), 7th edition (1908), pp. 517-519, and pp. 599, 600.

Let the current strength at A $= I_0$;
" " P $= I$;
" " Q $= I + dI$.

Also let resistance of unit length of conductor $= r$;
and " " " sheathing $= i$.

Then since the flow of electricity from any point to any other point close to it is from the point of higher to that of lower potential, and is equal to the difference of potential divided by the resistance separating the two points, therefore the current along AB at P is

$$\frac{(V+dV)-V}{rdx} = \frac{dV}{rdx} = I.$$

(The resistance of the wire PQ is rdx, because it varies *directly* as the length of the wire, but the resistance of the insulating sheath PQ is $\frac{i}{dx}$, because it varies *inversely* as the length.)

Hence the "leakage," that is, the current from the surface of the conductor between the points P and Q to the earth where the potential is zero, is

$$\frac{V-0}{\frac{i}{dx}} = \frac{Vdx}{i} = dI.$$

Hence,

$$\frac{dI}{dx} = \frac{V}{i}, \quad \text{or} \quad dI = \frac{Vdx}{i}.$$

but,

$$I=\frac{dV}{rdx}, \quad \text{or} \quad dI=\frac{d^2V}{rdx};$$

therefore,

$$\frac{d^2V}{rdx}=\frac{Vdx}{i}$$

therefore,

$$\frac{d^2V}{dx^2}=\frac{rV}{i}=m^2V,$$

where,

$$m^2=\frac{r}{i}, \quad \text{i.e.,} \quad m=\sqrt{\frac{r}{i}}$$

The solution of this differential equation, obtained by the well-known method,* is—

$$V=Ae^{mx}+Be^{-mx}, \quad - \quad - \quad - \quad [1]$$

and,

$$I=\frac{dV}{rdx}=\frac{m}{r}\left[Ae^{mx}-Be^{-mx}\right] \quad - \quad - \quad [2]$$

---

* See Boole's "Differential Equations," 2nd edition, Chapter IX., p. 194.

Now when $x=0$, then e^{mx} and e^{-mx} each equal 1, and V becomes the potential at **A**, *i.e.*, $V=V_0$; also $I=I_0$, and $I_0=\frac{V_0}{\sigma}$, or $\frac{V_0}{I_0}=\sigma$, hence we have,

$$\frac{V_0}{I_0}=\frac{A+B}{\frac{m}{r}\left(A-B\right)}=\frac{\frac{A}{B}+1}{\frac{m}{r}\left(\frac{A}{B}-1\right)}=\sigma,$$

from which

$$\frac{A}{B}=\frac{\sigma\frac{m}{r}+1}{\sigma\frac{m}{r}-1} \qquad - \quad - \quad - \quad [3]$$

In equation [2], calling l the length of the line, *i.e.*, putting $x=l$, we get

$$\text{Current } sent=I_s=\frac{m}{r}\left[Ae^{ml}-Be^{-ml}\right];$$

and from the same equation, by putting $x=0$,

$$\text{Current } received=I_r=\frac{m}{r}\left[A-B\right];$$

therefore,

$$\frac{I_s}{I_r}=\frac{Ae^{ml}+Be^{-ml}}{A-B}=\frac{\frac{A}{B}e^{ml}+e^{-ml}}{\frac{A}{B}-1};$$

but from equation [3] we have

$$\frac{A}{B}=\frac{\sigma\frac{m}{r}+1}{\sigma\frac{m}{r}-1};$$

therefore,

$$\frac{I_s}{I_r}=\frac{\dfrac{\sigma\frac{m}{r}+1}{\sigma\frac{m}{r}-1}e^{ml}+e^{-ml}}{\dfrac{\sigma\frac{m}{r}+1}{\sigma\frac{m}{r}-1}-1}=\frac{\left(\sigma\frac{m}{r}+1\right)e^{ml}+\left(\sigma\frac{m}{r}-1\right)e^{-ml}}{2}.$$

If $\sigma=0$, then

$$\frac{I_s}{I_r}=\frac{e^{ml}+e^{-ml}}{2}. \qquad \text{- - - (O)}$$

If we work out the value of $\frac{I_s}{I_r}$ from this equation, giving to ml the numerical value corresponding to that used in the example given on p. 42, then, since $ml=l\sqrt{rs}=\sqrt{\frac{R}{R_1}}$,

therefore

$$ml=\sqrt{\frac{360}{1000}}=\sqrt{\cdot 36}=\cdot 6,$$

therefore

$$\frac{I_s}{I_r}=\frac{(2{\cdot}7184)^{\cdot 6}+\dfrac{1}{(2{\cdot}7184)^{\cdot 6}}}{2}=\frac{1{\cdot}822+{\cdot}549}{2}=1{\cdot}185,$$

which is only $\frac{1}{2}$ per cent. higher than the result 1·180 given by the simple formula (L), p. 42.

37. That the simple formula is closely in accordance with the exact formula may be proved as follows:—

$$\frac{e^{l\sqrt{rs}}+e^{-l\sqrt{rs}}}{2}=$$

$$1+l\sqrt{rs}+\frac{(l\sqrt{rs})^2}{\underline{|2}}+\frac{(l\sqrt{rs})^3}{\underline{|3}}+\ldots 1-l\sqrt{rs}+\frac{(l\sqrt{rs})^2}{\underline{|2}}-\frac{(l\sqrt{rs})^3}{\underline{|3}}+\ldots$$

which, when $\sqrt{rs}$ is small, *i.e.*, where s is small (which in practice is usually the case), equals

$$\frac{2+(l\sqrt{rs})^2}{2}=1+\frac{l^2rs}{2},$$

which is formula (M), p 43.

38. If l, r, and s are varied in value, but in such proportions that the value of l^2rs or of $l\sqrt{rs}$, remains unaltered, then the value of $\frac{I_s}{I_r}$ is unaltered. Thus, for example, if $l=5$, $r=16$, and $s=4$, then

$$l\sqrt{rs}=15\sqrt{16\times 4}=15\times 8=120;$$

also, if $l=20$, $r=12$, and $s=3$, then

$$l\sqrt{rs}=20\sqrt{12\times 3}=20\times 6=120;$$

i.e., in both cases the attenuation is the same.

39. Equations (N), p. 43, and (O), p. 48, are true for alternating as well as for continuous currents, provided the circuit consists of conductor resistance and leakance only; for in this case the values of the conductor resistance and of the leakance do not vary as the current rises and falls, and consequently the rise and fall of the received current will coincide precisely as regards time, with the rise and fall of the transmitted current; that is to say, when the transmitted current is zero, then at the same moment the received current will be zero, and when the transmitted current reaches its maximum, then at that moment the received current will also have attained its maximum. The conditions are, however, different if inductance and capacity are present.

It has been shown that when r is inductive, its value becomes $\sqrt{r^2+L^2f^2}$, and if i, i.e. $\frac{1}{s}$, has capacity, its value becomes $\sqrt{s^2+C^2f^2}$.

We cannot, however, obtain the true attenuation of the current (*i.e.*, the ratio $\frac{I_s}{I_r}$ when this ratio has a maximum value) by substituting these modified values of r and i in the place of r and s as used in m in equations (N) and (O), for $\sqrt{r^2+L^2f^2}$ and $\sqrt{s^2+C^2f^2}$ represent the maximum values which the conductor resistance and the leakance attain, and these maximum values are not reached simultaneously (for actually $\sqrt{s^2+C^2f^2}$ lags behind $\sqrt{r^2+L_2f^2}$). There is, however, an intermediate value of the product of the two (when corrected for lag) which has a maximum value, and this maximum value will obviously make $\frac{I_s}{I_r}$ a maximum.

40. Now the differential expressions from which the general attenuation equation is obtained are those given on p. 45, viz.,

$$\frac{dV}{rdx}=I, \text{ or, } dV=Irdx, \text{ and } dI=\frac{Vdx}{i},$$

from which is obtained

$$\frac{d^2V}{dx^2}=\frac{rV}{i}.$$

To obtain, then, the required value of $\frac{I_s}{I_r}$, for i we have to substitute expression (K), p. 39, viz.,

$$\frac{1}{s \sin x^\circ - Cf \cos x^\circ};$$

and in regard to rdx, which is a differential part of the conductor alone, and has no reference to what resistances or combination of resistances, etc., precede or follow it, we know that (see "Electrical Testing," 7th Edition, p. 364, § 392) at any moment

$$dV=(Ir+E)dx,$$

where E is any back electromotive force which may be in r; and as

$$E=ILf$$

we have

$$dV=I(r+Lf)dx.$$

This becomes, when we express the rise and fall of dV,

$$dV = I(r \sin x + Lf \cos x)dx;$$

hence $\frac{r}{i}$ *i.e.*, rs, becomes

$$\frac{r \sin x^\circ + Lf \cos x^\circ}{\frac{1}{s \sin x^\circ - Cf \cos x^\circ}} = (r \sin x^\circ + Lf \cos x^\circ)(s \sin x^\circ - Cf \cos x^\circ)$$

$$= rs \sin^2 x^\circ - r\, Cf \sin x^\circ \cos x^\circ + s\, Lf \cos x^\circ \sin x^\circ - Lf\, Cf \cos x^\circ$$

$$= (s\, Lf - r\, Cf) \sin x^\circ \cos x^\circ + rs \sin^2 x^\circ - Lf\, Cf \cos^2 x^\circ = u; \quad \text{(P)}$$

All we have to do, therefore, is to find the maximum value of this expression, and this maximum will be the new value of $\frac{r}{i}$, *i.e.*, of rs, which we require.

Now

$$\frac{du}{dx} = (s\, Lf - r\, Cf)(\cos^2 x^\circ - \sin^2 x^\circ) + 2\, rs \sin x^\circ \cos x^\circ$$
$$+ 2\, Lf\, Cf \sin x^\circ \cos x^\circ = (s\, Lf - r\, Cf)(\cos^2 x^2 - \sin^2 x^2)$$
$$+ 2\,(rs + Lf\, Cf)(\sin x^\circ \cos x^\circ) = 0 \text{ at a maximum.}$$

Let

$$s\, Lf - r\, Cf = y, \quad rs + Lf\, Cf = z,$$

then

$$y\,(\cos^2 x^\circ - \sin^2 x^\circ) = -2z \sin x^\circ \cos x^\circ,$$

therefore

$$y\,(\cos^2 x^\circ - \sin^2 x^\circ)^2 = 4z^2 \sin^2 x^\circ \cos^2 x^\circ.$$

Also, let

$$\sin^2 x^\circ = A, \text{ then } \cos^2 x^\circ = 1 - A,$$

therefore

$$y^2\,(1-2A)^2 = 4z^2 A\,(1-A),$$

therefore

$$y^2\,(1+4A^2-4A) = 4z^2 A - 4z^2 A^2,$$

therefore

$$y^2 + y^2 4A^2 - y^2 4A = 4z^2 A - 4z^2 A^2,$$

therefore

$$y^2 + 4A^2(y^2+z^2) - 4A(y^2+z^2) = 0,$$

therefore

$$A^2 - A = -\frac{y^2}{4(y^2+z^2)},$$

therefore

$$A^2 - A + (\tfrac{1}{2})^2 = \tfrac{1}{4} - \frac{y^2}{4(y^2+z^2)} = \tfrac{1}{4}\,\frac{z^2}{(y^2+z^2)},$$

therefore

$$A - \tfrac{1}{2} = \pm\frac{z}{2\sqrt{y^2+z^2}},$$

therefore

$$A=\tfrac{1}{2}\pm\frac{z}{2\sqrt{y^2+z^2}}=\tfrac{1}{2}\left[1+\frac{z}{\sqrt{y^2+z^2}}\right]=\tfrac{1}{2}\left[\frac{\sqrt{y^2+z^2}+z}{\sqrt{y^2+z^2}}\right]$$

i.e.,

$$\sin^2 x^\circ=\tfrac{1}{2}\left[\frac{\sqrt{y^2+z^2}+z}{\sqrt{y^2+z^2}}\right], \quad - \quad - \quad [1]$$

and

$$\cos^2 x^\circ=1-\tfrac{1}{2}\left[\frac{\sqrt{y^2+z^2}+z}{\sqrt{y^2+z^2}}\right]=\frac{2\sqrt{y^2+z^2}-\sqrt{y^2+z^2}-z}{2\sqrt{y^2+z^2}}$$

$$=\tfrac{1}{2}\left[\frac{\sqrt{y^2+z^2}-z}{\sqrt{y^2+z^2}}\right]; \quad - \quad - \quad - \quad [2]$$

therefore

$$\sin^2 x^\circ \cos^2 x^\circ=\tfrac{1}{4}\left[\frac{\sqrt{y^2+z^2}+z}{\sqrt{y^2+z^2}}\times\frac{\sqrt{y^2+z^2}-z}{\sqrt{y^2+z^2}}\right]=\tfrac{1}{4}\,\frac{y^2}{y^2+z^2},$$

therefore

$$\sin x^\circ \cos x^\circ=\frac{y}{2\sqrt{y^2+z^2}}; \quad - \quad - \quad [3]$$

substituting the values [1], [2], and [3], in expression (P), p, 52, viz.:

$$(sLf-rCf)\sin x^\circ \cos x^\circ+rs\sin^2 x^\circ-Lf\,Cf\cos^2 x^\circ,$$

we get

$$y\frac{y}{2\sqrt{y^2+z^2}}+rs\,\tfrac{1}{2}\left[\frac{\sqrt{y^2+z^2}+z}{\sqrt{y^2+z^2}}\right]-Lf\,Cf\,\tfrac{1}{2}\left[\frac{\sqrt{y^2+z^2}-z}{\sqrt{y^2+z^2}}\right]$$

$$=\frac{1}{2\sqrt{y^2+z^2}}[y^2+rs(\sqrt{y^2+z^2}+z)-Lf\,Cf\,(\sqrt{y^2+z^2}-z)]$$

$$=\frac{1}{2\sqrt{y^2+z^2}}[y^2+(rs-Lf\,Cf)\sqrt{y^2+z^2}+z(rs+Lf\,Cf)]$$

$$=\frac{1}{2\sqrt{y^2+z^2}}[y^2+(rs-Lf\,Cf)\sqrt{y^2+z^2}+z^2]$$

$$=\tfrac{1}{2}\,[\sqrt{y^2+z^2}+rs-Lf\,Cf\,]$$

$$=\tfrac{1}{2}\,[\sqrt{(sLf-rCf\,)^2+(rs+Lf\,Cf\,)^2}+rs-Lf\,Cf\,]$$

$$=\tfrac{1}{2}\,[\sqrt{s^2L^2f^2+r^2C^2f^2-2rsLfCf+r^2s^2+L^2f^2C^2f^2+2rsLfCf}+rs-LfCf$$

$$=\tfrac{1}{2}\,[\sqrt{s^2L^2f^2+r^2C^2f^2+r^2s^2+L^2f^2\,C^2f^2}+rs-Lf\,Cf\,]$$

$$=\tfrac{1}{2}\,[\sqrt{(r^2+L^2f^2)(s^2+C^2f^2)}+rs-LCf^2],$$

which is the expression required.

41. The root of this quantity (corresponding to $\sqrt{\frac{r}{i}}$, or $\sqrt{rs}$) is called the "Attenuation Constant," and is known as "β," *i.e.*,

$$\beta=\sqrt{\tfrac{1}{2}\,(\sqrt{(r^2+L^2f^2)(s^2+C^2f^2)}+rs-LCf^2)}. \qquad (Q)$$

Loading (Inductance) Coils for Telephonic Transmission

42. In reference to equation (O), p. 48, *i.e.*,

$$\frac{I_s}{I_r}=\frac{e^{ml}+e^{-ml}}{2},$$

in order to make $\frac{I_s}{I_r}$ as small as possible, *i.e.*, to make I_r as near to I_s as possible, m, *i.e.*, $\sqrt{rs}$, must be made as small as possible; that is to say, in the case where alternating currents are concerned, β must be made as small as possible, which would be the case if it were possible to make LC^2f^2 equal to $\sqrt{(r^2+L^2f^2)(s^2+C^2f^2)}+rs$. To do this, however, would mean adding to the conductor of the cable sufficient inductance to bring about this result.

43. Inductance, however, cannot be obtained independent of resistance, and, so far, practical experience has shown that the best constructed inductance coils have for every 100mH of inductance a resistance (with a steady current) of not less than 6Ω, which largely discounts the effectiveness of the inductance.

CHAPTER VII

PRACTICAL FORMULÆ

(TELEPHONIC TRANSMISSION)

44. Formula (Q), p. 55, is not a convenient one for general use in connection with telephonic transmission problems, as the quantities involved are in c.g.s. units, and telegraph engineers never deal with "henries," "farads," and "leakances," but always with "millihenries," "microfarads," and "insulation resistances"; the term "leakance" conveys absolutely no significant practical meaning to a telegraph engineer, whereas "insulation resistance" is thoroughly well understood; adherence to the terms referred to is not only liable to create confusion, but may be a source of endless numerical mistakes when making practical calculations.

Looped Circuits

45. Although telephonic circuits in most cases are formed of loops, yet for the purpose of mathematical analysis they may be regarded as being single wire circuits earthed at both ends, and the unit values of r, L, s, and C which would be used in a formula would be those of a single wire earthed circuit, although the latter is actually a loop. An insulated

conductor earthed at both ends is, generally speaking, exactly equivalent to two similar conductors joined together so as to form a loop. This equivalency holds good for either low or high frequencies, provided the conductors are individually surrounded by a liquid conductor and not by a metallic sheathing.

In the case of low frequencies, such as occur in the case of ordinary telegraphic signals, the effect of even a metallic sheathing is extremely small, as has been proved by experiment, and as is well known. Where, however, telephonic frequencies are concerned, the effect of a metallic sheathing, as in the case of a submarine cable, is very noticeable, hence the "β" of two looped cores in a single cable is different from that of two similar looped cores contained in two separate cables, as in one case the iron sheathing surrounds the two looped cores as a whole, and in the other case each of the looped cores is individually surrounded. Experiments made on cables 80 knots long, of the Post Office type, showed that the effective "β" was about 20 per cent. greater (*i.e.*, the efficiency correspondingly less) in the case of the looped separate cables than in the case of the looped cores in one cable; also it was found that a single core cable earthed at both ends was similarly less efficient than two cores looped in one cable.

46. On the face of it, it might appear comparatively easy to solve the general equation (Q), p. 55, by giving true values to r, f, L, s, and C, and then working out the result by direct calculation. Actually, however, this is not by any means the case, as the numerical value of the part

$$\sqrt{(r^2+L^2f^2)(s^2+C^2f^2)}+rs$$

of the equation is, in the majority of cases, so very nearly equal to the numerical value of the part

$$LCf^2,$$

that unless the terms are worked out to a great number of places of figures, and the square root value is also very closely calculated to a large number of places of figures, the true value of β, even to a moderate degree of accuracy, cannot be ensured. In fact, the formula

$$\beta = \sqrt{\tfrac{1}{2}(\sqrt{(r^2+L^2f^2)(s^2+C^2f^2)}+rs-LCf^2)},$$

as it stands is almost useless from a practical point of view.

47. In order to translate the formula into one which can readily be handled, let

L = inductance in mH.
C = capacity in μH,
o = insulation resistance, in O, at telephonic frequency.

Then for L in the formula we must put $\frac{L}{1000}$, and for C we must put $\frac{C}{1{,}000{,}000}$, also $s=\frac{1}{o}$. For telephonic frequency, it has been found that $f=5000$, approximately.

If we insert these values in the formula in question, the latter becomes very approximately

$$\beta = \frac{\sqrt{C}}{20}\sqrt{\sqrt{r^2+(5L)^2}-5L+\frac{200r}{oC}+\cdot 000128\sqrt{r^2+(5L)^2}} \quad [1].$$

The value of oC for ordinary gutta-percha insulated cores may be taken to be 12,500 per knot, approximately; that is to say, we get (taking C, r, and L all to be *per knot* values) — for all values

a general equation of L & r

$$\beta = \frac{\sqrt{C}}{20}\sqrt{\sqrt{r^2+(5L)^2} - 5L + \cdot 016r + \cdot 000128\sqrt{r^2+(5L)^2}} \quad [2].$$

This formula can be readily used and gives accurate results, provided the value of $\sqrt{r^2+(5L)^2}$ is worked out to not less than six or seven places of figures, and provided the expression $\cdot 000128\sqrt{r^2+(5L)^2}$ is not assumed to be so small as to be negligible.

48. In the formula, r is the conductor resistance plus the resistance of the inductance (loading) coils, and should more correctly be given as

$$r + \rho,$$

where ρ is the resistance of the coils.

In the case of well constructed coils, as was pointed out in § 43, p. 56,

$$r = L \times \cdot 06$$

approximately, *i.e.*, for example, an inductance coil having an inductance of 50mH will have a resistance of

$$50 \times \cdot 06 = 3\omega.$$

49. Since for ordinary gutta-percha insulated cores

insulation $\quad oC = 12{,}500$ per knot,

we can see that a core of this kind having a capacity of, say, ·280mC per knot, would have a value o of

$$o = \frac{12,500}{\cdot 280} = 44,600\text{O per knot,}$$

at telephonic frequency.

50. This value of o is, of course, very much lower than the insulation per knot with a steady current after, say, one minute's electrification, the insulation in the latter case being of the order of several hundred, or even thousand, MO.

51. As is well known, the shorter the electrification time the less is the observed insulation resistance of a cable core, and inasmuch as average telephonic frequency is about 750 per second, the value of o may, it seems probable, approximate to that which would be obtained by a steady current insulation test if it were possible to obtain a measurement a fraction of a second after the current were applied to the core. It may be stated, however, that, so far, no relation between the insulation resistance with a steady current and the insulation resistance with a current of high frequency has been discovered; indeed, it has been found that a core which has a steady current insulation resistance of, say, 1000MO per knot after one minute's electrification, may have a telephonic frequency insulation of 50,000O per knot, whilst another core may have a steady current insulation resistance of, say, 500MO per knot, whilst the telephonic frequency resistance may be 75,000O per knot.

52. Formula [2], p. 60, is a general one, and is applicable for all values of L and r. In most cases, however, L (in the case of a loaded cable), is greater in numerical

value than r; we may, therefore, in all cases when L much exceeds r, employ a simplified formula, as follows:—

If we expand $\sqrt{r^2+(5L)^2}$ and neglect the small terms, we find that $\sqrt{r^2+(5L)^2}=5L+\frac{r^2}{10L}$, approximately, so that

$$\beta=\frac{\sqrt{C}}{20}\sqrt{5L+\frac{r^2}{10L}-5L+{\cdot}016r+{\cdot}000128\left(5L+\frac{r^2}{10L}\right)}$$

$$=\frac{\sqrt{C}}{20}\sqrt{\frac{r^2}{10L}+{\cdot}016r+{\cdot}000128\times 5L}$$

(since $\frac{r^2}{10L}$ is small compared with $5L$),

$$=\frac{\sqrt{C}}{20\sqrt{10}}\sqrt{\frac{r^2}{L}+{\cdot}16r+{\cdot}0064L} \quad - \quad - \quad - \quad [3]$$

$$=\sqrt{\frac{C}{L}}\cdot\frac{\sqrt{r^2+{\cdot}16rL+{\cdot}0064L^2}}{20\sqrt{10}}=\sqrt{\frac{C}{L}}\cdot\frac{r+{\cdot}08L}{63{\cdot}2}.$$

If we substitute

$r+\rho$, that is, $r+{\cdot}06L$,

for r, then

$$\beta=\frac{r+L({\cdot}06+{\cdot}08)}{63{\cdot}2}\sqrt{\frac{C}{L}} \quad - \quad - \quad [4]$$

$$=\frac{r+{\cdot}14L}{63{\cdot}2}\sqrt{\frac{C}{L}} \quad - \quad - \quad - \quad [5]$$

The value of β in equation [5] is susceptible of a minimum* by variation of L, and this is the case when

$$r = \cdot 14L, \text{ or } L = \frac{r}{\cdot 14} = 7\cdot 14r; \qquad [6]$$

so that we get

$$\beta = \frac{2r}{63\cdot 2}\sqrt{\frac{C \times \cdot 14}{r}} = \frac{\sqrt{\cdot 14}}{31\cdot 6}\sqrt{Cr} = \cdot 01183\sqrt{Cr}. \quad [7]$$

Equations [6] and [7], then, give the conditions for making β as low as possible.

Example:—What "attenuation constant" can be obtained on a cable insulated with ordinary gutta-percha, and having a conductor resistance of 6·7Ο per knot and capacity ·280μF per knot, and what should be the loading inductance?

1 Knot = 6080.27 ft.

Answer.

$$L = 7\cdot 14 \times 6\cdot 7 = 48\text{mH per knot.}$$

$$\beta = \cdot 01183\sqrt{\cdot 280 \times 6\cdot 7} = \cdot 0162 \text{ per knot.}$$

53. It should be distinctly understood that formula [7] only holds good provided that formula [6] is also satisfied. In cases where L is not in accordance with formula [7], then in order to obtain β, formula [5] must

* $\beta = \frac{r + \cdot 14L}{63\cdot 2}\sqrt{\frac{C}{L}} = \frac{r + \cdot 14L}{\sqrt{L}} \cdot \frac{\sqrt{C}}{63\cdot 2}$; let $u = \frac{r + \cdot 14L}{\sqrt{L}}$; therefore, $\frac{du}{dL} = \frac{1}{L}\left(\sqrt{L} \times \cdot 14 - (r + \cdot 14L)\frac{1}{2\sqrt{L}}\right) = 0$ at a minimum; therefore, $\sqrt{L} \times \cdot 14 = (r + \cdot 14L)\frac{1}{2\sqrt{L}}$, or $2L \times \cdot 14 = r + \cdot 14L$; therefore $r = \cdot 14L$.

be used; but, as before pointed out, this formula is only correct when L is not less in numerical value than r.

54. In reference to the quantity ·08 in formula [4], it should be pointed out that the value ·08 (which is a value dependent upon the insulation resistance at mean telephonic frequency, and also upon the capacity), is correct for ordinary gutta-percha, but by the use of a suitable dielectric its value may be very considerably diminished, in which case, as is obvious from formula [4], the value of β (upon which the speaking distance depends) can be very materially diminished.

55. For the purpose of obtaining the value of β with various values of the insulation resistance at telephonic frequency, formula [3] may be utilised. Substituting in this formula, $r + \cdot 06L$ for r (pp. 60 and 62), and also substituting for the quantity $\cdot 16r$ the value $\frac{2000r}{oC}$, to which it corresponds (see formulæ [1] and [2], p. 60), we get

$$\beta = \frac{\sqrt{C}}{20\sqrt{10}}\sqrt{\frac{(r+\cdot 06L)^2}{L}+\frac{2000}{oC}(r+\cdot 06L)+\cdot 0064L}$$

$$= \frac{\sqrt{C}}{20\sqrt{10}}\sqrt{\frac{r^2}{L}+\cdot 0036L+\cdot 12r+\frac{2000r}{oC}+\frac{120L}{oC}+\cdot 0064L}$$

$$= \frac{\sqrt{C}}{20\sqrt{10}}\sqrt{\frac{r^2}{L}+\left(\cdot 01+\frac{120}{oC}\right)L+\cdot 12r+\frac{2000r}{oC}}.$$

The value of β in this equation is susceptible of a minimum by variation of L, *i.e.*, we have to make

$\frac{r^2}{L}+\left(\cdot01+\frac{120}{oC}\right)L$ a minimum; this will be the case when $L=\frac{r}{\sqrt{\cdot01+\frac{120}{oC}}}$, in which case

$$\beta=\frac{\sqrt{Cr}}{63\cdot2}\sqrt{2\sqrt{\cdot01+\frac{120}{oC}}+\cdot12+\frac{2000}{oC}}.$$

If in this formula we give the value 12500 to oC (p. 60) we get

$$\beta=\frac{\sqrt{Cr}}{63\cdot2}\sqrt{2\sqrt{\cdot01+\frac{120}{12500}}+\cdot12+\frac{2000}{12500}}.$$

$$=\frac{\sqrt{Cr}}{63\cdot2}\sqrt{\cdot28+\cdot12+\cdot16}=\frac{\sqrt{Cr}}{63\cdot2}\sqrt{\cdot56}=\cdot01183\sqrt{Cr},$$

i.e., formula [7], p. 63.

56. If it were possible to reduce the leakance to zero, *i.e.*, to make the insulation, o, equal to ∞, then the formula would become

$$\beta=\frac{\sqrt{Cr}}{63\cdot2}\sqrt{\cdot2+\cdot12}=\frac{\sqrt{Cr}}{63\cdot2}\sqrt{\cdot32}=\cdot008951\sqrt{Cr}, \quad - \quad [8]$$

a result 32 per cent. beyond that given by ordinary gutta-percha formula [7] (p. 63). This amount then represents the maximum gain which would be possible if we assume that the insulation resistance could be

made practically infinite. Actually this would not mean making o excessively large; thus if we increased o tenfold, that is to say, if it made the value of oC to be 125000 instead of 12500 (p. 59), then we should have

$$\beta = \frac{\sqrt{Cr}}{63{\cdot}2}\sqrt{2\sqrt{{\cdot}01 + \frac{120}{125000}} + {\cdot}12 + \frac{2000}{125000}}$$

$$= \frac{\sqrt{Cr}}{63{\cdot}2}\sqrt{{\cdot}21 + {\cdot}12 + {\cdot}016} = \frac{\sqrt{Cr}}{63{\cdot}2}\sqrt{{\cdot}346} = {\cdot}009307\sqrt{Cr},$$

a result 27 per cent. beyond that given by ordinary gutta-percha, and, moreover, a result which has actually been obtained in the most recent loaded cables manufactured and laid.

Any further reduction in the value of β could only be obtained by diminishing the value of the resistance of the inductance coils, *i.e.*, by reducing the quantity ·06 (p. 60), but, so far, no success has been obtained in this direction, all important as it is, and it hardly seems probable that any material reduction is likely to be obtained.

Inductance of an Unloaded Cable

57. It is sometimes assumed that the inductance of an unloaded cable, being very small, is negligible. This, however, is not the case, as can be seen from formula [1], p. 59. To take a practical example :—

Suppose $r = 6{\cdot}70$, $C = {\cdot}280\mu\mathrm{F}$, and $L = 0\mathrm{H}$, then formula [2], p. 60, becomes

$$\beta = \frac{\sqrt{C}}{20}\sqrt{r + {\cdot}016\ r}$$

(since $\cdot000128\sqrt{r^2+(5L)^2}$, in this case is negligible).

Inserting the value of r, we get

$$\beta = \frac{\sqrt{\cdot280}}{20}\sqrt{6{\cdot}807} = {\cdot}069.$$

Next, suppose that $L = 1$**mH**, then,

$$\beta = \frac{\sqrt{\cdot280}}{20}\sqrt{\sqrt{6{\cdot}7^2+25}-5+{\cdot}107} = {\cdot}049\,;$$

i.e., the value of β has been lowered from ·069 to ·049, or 30 per cent., by the addition of an inductance of 1**mH** only.

58. By means of formulæ [5] (p. 62) and [2] (p. 60) it is possible to determine, by direct experiment, what the value of L is in any particular cable or wire. Thus, to take the case of the Anglo-French loaded cable, which has a conductor resistance of 6·7**O** and a capacity of ·280μ**F** per knot, it was found by experiment that the loaded cable, compared with a similar cable unloaded, had a relative value of 3·2 to 1, the loading inductance being 50**mH** per knot.

From formula [5], p. 62, we have

$$\beta=\frac{6{\cdot}7+{\cdot}14\times 50}{63{\cdot}2}\sqrt{\frac{{\cdot}280}{50}}={\cdot}0162.$$

From formula [2], p. 60, we have (since we know that L will be small),

$${\cdot}0162\times 3{\cdot}2=\frac{\sqrt{{\cdot}280}}{20}\sqrt{\sqrt{6{\cdot}7^2+(5L)^2}-5L+{\cdot}016\times 6{\cdot}7},$$

therefore

$$\left(\frac{{\cdot}0162\times 3{\cdot}2\times 20}{{\cdot}529}\right)^2-{\cdot}1072=\sqrt{6{\cdot}7^2+(5L)^2}-5L,$$

therefore

$$3{\cdot}73+5L=\sqrt{6{\cdot}7^2+(5L)^2},$$

therefore

$$13{\cdot}9+25L^2+37{\cdot}3L=44{\cdot}89+25L^2,$$

therefore

$$37{\cdot}3L=30{\cdot}99,$$

or

$$L=\frac{30{\cdot}99}{37{\cdot}3}={\cdot}83\text{mH}.$$

59. The number 3·2 in the foregoing case (§ 58) may be termed the "*Improvement factor,*" or to adapt the term to modern nomenclature, the "*Improvance,*" and for any cable, will be the ratio between formulæ [2], p. 60, and [5], p. 62, *i.e.*,

$$\text{Improvance} = \frac{\frac{\sqrt{C}}{20}\sqrt{\sqrt{r^2+(5L)^2}-5L+\cdot016r}}{\cdot01183\sqrt{Cr}}$$

$$= 4\cdot23\sqrt{\sqrt{1+\left(\frac{5L}{r}\right)^2}-\frac{5L}{r}+\cdot016} \quad - \quad - \quad [9]$$

60. An exact determination of this "Improvance" by calculation can only be obtained if it is possible to determine the precise value of L (the normal inductance in an unloaded cable); for even if L is small it cannot (see § 57) be regarded as an unimportant quantity. Thus, for example, if we have $r=6\cdot7\Omega$, and $L=\cdot83$**mH**, we get

$$\text{Improvance} = 4\cdot23\sqrt{\sqrt{1+\left(\frac{5\times\cdot83}{6\cdot7}\right)^2}-\frac{5\times\cdot83}{6\cdot7}+\cdot016} = 3\cdot20.$$

If, however, we take $L=1$**mH**, instead of $\cdot83$**mH**, then

$$\text{Improvance} = 4\cdot23\sqrt{\sqrt{1+\left(\frac{5}{6\cdot7}\right)^2}-\frac{5}{6\cdot7}+\cdot016} = 3\cdot04.$$

i.e., we cannot regard ·83 as being approximately equivalent to 1.

61. From formula [9] (above), we can see that the possible improvement by loading, diminishes as the conductor resistance diminishes. Thus if we assume $L=1$**mH**, we get

$$\text{Improvance} = 4\cdot23\sqrt{\sqrt{1+\left(\frac{5}{r}\right)^2}-\frac{5}{r}+\cdot016}\,;$$

and if we give r the various values 8, 7, 6, 5, 4, 3, 2 and 1, then the corresponding improvances will be

rO - -	8	7	6	5	4	3	2	1
Improvance	3·2	3·1	2·9	2·8	2·6	2·3	1·9	1·4.

Thus the improvance for a 1O conductor is but one-half that for a 5O conductor.

62. It should be noted that the improvance is apparently independent of capacity; but this, strictly speaking, is not the case, since the quantities ·016 [2] (p. 60) and ·01183 [7] (p. 63), both contain C as a factor; but to what extent variation of C would affect these values is uncertain, since the value of o, which is also a factor of the quantities in question, may possibly be different for different values of C; experimental data on the subject, up to the present, are wanting.

Loaded Aerial Lines

63. Experience with aerial lines has shown that the improvance is not so great as with underground or submarine circuits, but whether this is simply due to the value of C being low is uncertain. The actual value of C for an aerial wire is, of course, very much less than that for a submarine cable, the ratio being on an average about 1 to 20; this, irrespective of the value of o in formula [1] (p. 59), would largely increase the value of the quantity ·016 in formula [2] (p. 60), and make β correspondingly high; but on the other hand the insulation resistance with telephonic frequency in the case of aerial lines, when the normal insulation is high, may also be high, and this may very largely discount the low value of C, and considerably reduce the quantity ·016.

CHAPTER VIII

REFLECTION

64. Whenever a circuit (traversed by an alternating current) in which capacity, inductance, etc., is involved, is not uniform along its length, as, for example, when a submarine cable is connected on to a land line, or even when it is terminated through an ordinary ohmic resistance, *reflection* takes place as a consequence of this absence of uniformity. This reflection (which is only noticeable if the circuit is of considerable magnitude) has the effect of reducing the received current beyond the value which might be expected if the reduction were an ohmic effect only.

To calculate the reflection, the latter may be regarded as a current set up at the point at which the change of uniformity has taken place, this current being set up by the incident current at the point in question.

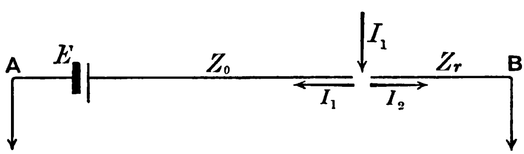

Fig. 23.

Let I_1 (Fig. 23) be the incident current at the point of change, then this current will split up into two portions, I_1 and I_2, and the relative values of I_1 and I_2 will be in proportion to the impedances (effective

resistances), Z_o and Z_r, through which I_1 and I_2 flow, *i.e.*, we have

$$I_1 = I_1 \frac{Z_r}{Z_r + Z_o},$$

$$I_2 = I_1 \frac{Z_o}{Z_r + Z_o}.$$

Now, the resultant or reflected current, I_r (at B), will be the difference between I_1 and I_2, since these flow in opposite directions, *i.e.*,

$$I_r = I_1 - I_2 = I_1\left(\frac{Z_r}{Z_r + Z_o} - \frac{Z_o}{Z_r + Z_o}\right) = I_1 \frac{Z_r - Z_o}{Z_r + Z_o} \qquad \text{- (R)}$$

The current I_t, which will actually flow through Z_r, will be the difference between I_1 and I_r, since we must have

$$I_1 = I_r + I_t,$$

that is,

$$I_t = I_1 - I_r;$$

hence

$$I_t = I_1\left(1 - \frac{Z_r - Z_o}{Z_r + Z_o}\right) = I_1 \frac{2\,Z_o}{Z_r + Z_o}. \qquad \text{- (S)}$$

The quantity $\frac{2\,Z_o}{Z_r + Z_o}$ which indicates the ratio of I_t to I_1 is called the "coefficient of transmission" (m) whilst $\frac{Z_r - Z_o}{Z_r + Z_o}$ (which is equal to $1 - m$) is called the "coefficient of reflection," since from (R)

$$I_r = I_1 (1 - m).$$

From (S) it can be seen that if $Z_r = Z_o$, then $I_t = I_1$; *i.e.*, there is no reflection. Also, it can be seen that if $Z_r = 0$ then

$$I_t = 2\,I_1.$$

65. Reflection takes place if, as stated, the end of a submarine cable is put to earth (or looped) through a resistance, instead of being direct to earth (or looped direct). For example, in an experiment made on a cable, the two cores of which were looped at their further ends through resistances of various values, the following results were obtained:—Z_o being the normal impedance of the cable, *i.e.*, the effective resistance when the ends of the cores were looped together without additional resistance in circuit.

Terminal Impedance (Resistance) (Z_r)	$I_r \times \left(\frac{2 Z_o}{Z_o + Z_r}\right)$	Observed Received Current
O	mA	mA
0	4·16	4·02
100	3·60	3·57
200	3·18	3·20
300	2·90	2·87
400	2·57	2·62
500	2·45	2·38
600	2·24	2·22
650	2·08	2·08
700	2·0	2·04
800	1·86	1·90
900	1·74	1·80
1,000	1·64	1·65
2,000	1·02	1·13

The calculated and observed results are, it will be noticed, in close agreement, allowance being made for experimental error.

CHAPTER IX

INSTRUMENTAL MEASUREMENT OF AN ALTERNATING CURRENT

66. The power of a current of doing work, its heating effect, for example, in the case of a hot air ammeter, or the "torque" in the case of an ammeter of the non-polarised type, is in proportion to the square of the value of the current.*

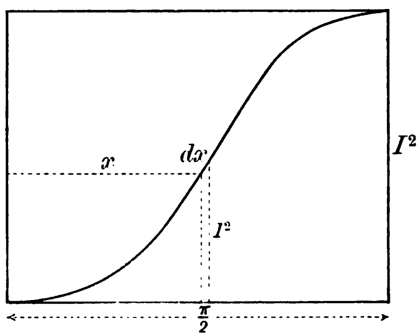

Fig. 24.

* See § 68, p. 77. In the case of a non-polarised instrument, where we have a fixed coil acting upon a movable coil, both in the same circuit, or a coil acting upon an iron core magnetised by the coil, then when the current is doubled we have a doubled effect in each of the two elements (a coil and coil in one case and a coil and iron core in the other case), consequently we have a doubled effect acting upon a doubled effect, *i.e.*, a quadrupled effect as a total.

If, then, I^2 represents the square of the current flowing at the end of the part x of the total $\frac{1}{4}$ period $\frac{\pi}{2}$, the heating effect at that moment will be $I^2\,dx$, and the total heating effect, H_a over the total $\frac{1}{4}$ period $\frac{\pi}{2}$, will be

$$H_a = \int_0^{\frac{\pi}{2}} I^2\,dx;$$

but

$$I = I_a \sin x,$$

or

$$I^2 = I_a^2 \sin^2 x,$$

therefore

$$H_a = \int_0^{\frac{\pi}{2}} I_a^2 \sin {}^2x\,dx$$

$$= I_a^2 \left[\tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) \right]_0^{\frac{\pi}{2}} = I_a^2\,\tfrac{1}{2}\left(\frac{\pi}{2} - \tfrac{1}{2}\sin\pi\right) - 0$$

$$= I_a^2\,\tfrac{1}{2}\cdot\frac{\pi}{2} = I_a^2\frac{\pi}{4}.$$

But the heating effect H_c of a continuous current I_c is $H_c = I_c^2\frac{\pi}{2}$,

therefore

$$H_a : H_c :: I_c^2\frac{\pi}{2} : I_a^2\frac{\pi}{4}$$

$$:: I_c^2 \quad : \frac{I_a^2}{2}$$

$$:: I_c^2 \quad : (\cdot 707\,I_a)^2.$$

That is to say, the effective value of an alternating current is ·707 of its maximum value.

67. The conditions for measuring an alternating voltage on a voltmeter are the same as those for measuring an alternating current on an ammeter.

Heating Effect of a Current

68. If we take a one-cell Daniell battery of practically no resistance and join it through a high resistance, a certain amount of heat will be generated in the latter, and a certain amount of copper will be thrown down on the copper plate, and a certain amount of zinc dissolved from the zinc plate in the cell. If, now, we join a second cell in the circuit, we shall have doubled the electromotive force, and consequently the current flowing in the circuit will be doubled, but the amount of heat generated on the resistance will be more than doubled, it will be quadrupled. The reason of this is as follows:—It is evident that since we have doubled the current flowing in the circuit, the amount of current passing in the first cell is doubled, and therefore the quantity of copper thrown down on the copper plate, and the quantity of zinc dissolved in a given time, is doubled. But as we have two cells in the circuit, the total amount of copper thrown down and of zinc dissolved is four times what it was at first; and as a battery produces force by the expenditure of material (decomposition of sulphate of copper into copper and sulphuric acid, etc.), just as force is produced by the combustion of coal in a steam engine, therefore by

doubling the electromotive force we have quadrupled the amount of energy—that is, the amount of heat developed in the wire. The heat generated, in fact, varies as the *square* of the current flowing in the circuit. At first sight this law seems paradoxical, and to many it is a matter of great difficulty to understand it. The following explanation, therefore, may make the matter clearer.

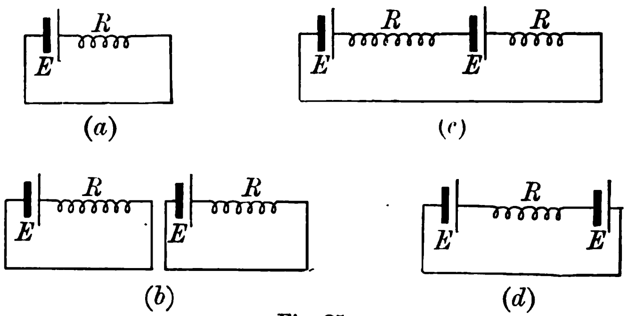

Fig. 25.

In Fig. 25, (*a*) shows a single cell battery, *E*, joined up in circuit with a resistance, *R*; we have therefore a certain amount of heat generated in the resistance, *R*. (*b*) shows *two* such circuits; in this case it is evident that the *total* amount of heat which is being generated by the two circuits is *twice* that generated by the single circuit. If now we combine the two circuits, as shown by (*c*), we still have the same condition of affairs as that which existed when the arrangement was that shown by (*b*)—in other words, the resistances, *R*, *R*, remain heated to the same extent. But if, finally, we remove one of the resistances, *R*, so that the circuit is that shown by (*d*), then if we can suppose in the

first place that no change has taken place in the strength of the current, it is still evident that the heat which was generated and distributed over two resistances, R, R, is now distributed over one only, and consequently the change must have caused the single resistance, R, to have become heated to twice the extent that it was originally. But further still, the change in the resistance of the circuit from $2R$ to R must have doubled the strength of the current, and for this reason also the amount of heat developed in R must have become doubled. It therefore follows that the actual development of heat, under the conditions shown by (d), must be four times that which occurs when the circuit is that shown by (a).

We see, then, that the quantity of heat developed in a resistance varies as the *square* of the current passing, but the heat developed varies also as the resistance overcome; hence we have Joule's law,

$$H = I^2R,$$

where H is the number of heat units developed.

Since we have

$$I = \frac{E}{R},$$

where E is the electromotive force producing the current, therefore

$$H = \frac{E^2}{R^2} \times R = \frac{E^2}{R} = EI.$$